Constrained current algebras and $g/u(1)^d$ conformal field theories.

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Abstract

Operator quantization of the WZNW theory invariant with respect to an affine Kac-Moody algebra $\hat{g}$ with constrained $\hat{u}(1)^d$ currents is performed using Dirac's procedure.Upon quantization the initial energy-momentum tensor is replaced by the $g/u(1)^d$ coset construction.The $\hat{su}(2)$ WZNW theory with a constrained $\hat{u}(1)$ current is equivalent to the $su(2)/u(1)$ conformal field theory.

1.In this article we study the connection between the WZNW theory [1, 2] , associated with a simply-laced algebra g, and the $g/u(1)^d$ conformal field theory, whose energy-momentum tensor K_g is constructed via Goddard,Kent and Olive coset method [3]

$$K_g = L_g - L_{u(1)^d} \tag{1}$$

These theories for $d = r$,where $r = rank\, g$, are closely related. In [4] Fateev and Zamolodchikov found that all the fields of the $su(2)$ WZNW theory can be expressed in terms of the $su(2)/u(1)$ parafermions and a free boson.This relation and its generalization to the case of arbitrary algebra g were used for computation of modular invariant partition functions of $su(2)/u(1)$ [5] and $g/u(1)^r$ [6] theories.

The object of this article is to present a new method of constructing the $g/u(1)^d$,$1 \leq d \leq r$, conformal theory from the WZNW theory invariant with respect to an affine Kac-Moody (KM) algebra $\hat{g}$. It is based on the operator quantization of the WZNW theory with constraints

$$H_A(z) \approx 0, \tag{2}$$

where $H_A(z), A = 1,\ldots,d$, are generators of $\hat{u}(1)^d$ subalgebra of the algebra $\hat{g}$.Here and in what follows we treat only the holomorphic part. We quantize the system in terms of the modes of $H_A(z)$, which form an algebra of first and second

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class constraints.The gauge is fixed by an extention of this algebra.The theory can then be quantized by replacing the initial operators by the operators which are constructed using Dirac's procedure [7]. This method can be generalized to other coset conformal theories.

2.The quantization will use the bosonic construction of the KM currents and Virasoro generator L_g of the WZNW theory in terms of the Fubini-Veneziano fields

$$\varphi_s^j(z) = q_s^j - ip_s^j log z + i \sum_{n \neq 0} \frac{a_{sn}^j}{n} z^{-n}, \tag{3}$$

where $s = 1, \ldots, r$, $j = 1, \ldots, k$, k is the level of the representation of the KM algebra $\hat{g}$ and

$$[q_s^j, p_t^l] = i\delta^{jl}\delta_{st}, \qquad [a_{sm}^j, a_{tn}^l] = m\delta^{jl}\delta_{st}\delta_{m+n,0}. \tag{4}$$

We denote by Γ the algebra (4). The bosons (3) have the two-point functions

$$< 0|\varphi_s^j(z)\varphi_t^l(w)|0 > = -\delta^{jl}\delta_{st}\ln(z-w). \tag{5}$$

The bosonic construction of the currents (2) at level k is given by

$$H_A(z) = \sum_{j=1}^{k} i\partial_z \varphi_A^j(z). \tag{6}$$

It is convenient to decompose the constraints into modes

$$H_A(z) = H_{Ao}z^{-1} + \sum_{n \neq 0} H_{An}z^{-n-1},$$

$$H_{Ao} = \sum_{j=1}^{k} p_A^j, \qquad H_{An} = \sum_{j=1}^{k} a_{An}^j, \tag{7}$$

and consider an equivalent set of constraints

$$H_{Ao} \approx 0, \qquad H_{An} \approx 0. \tag{8}$$

The operators H_{Ao}, H_{An} obey the KM algebra

$$[H_{Ao}, H_{Bo}] = [H_{Ao}, H_{Bn}] = 0, \qquad [H_{Am}, H_{Bn}] = km\delta_{AB}\delta_{m+n,0}. \tag{9}$$

The commutator relations (9) tell us that the operators H_{Ao} are first class constraints and the H_{An} are second class ones. We take as gauge condition

$$q_A \equiv \sum_{i=1}^{k} q_A^i \approx 0. \tag{10}$$

The operators q_A commute with H_{Am} and

$$[q_A, H_{Bo}] = ik\delta_{AB}. \tag{11}$$

It follows from (9) and (11) that the constraints

$$\Phi = (H_{Ao}, H_{An}, q_A) \tag{12}$$

are second class ones.

Let g be simply-laced. In this case the KM currents $J(z) = (E_\alpha(z), H_s(z))$, where α are roots of $g, s = 1, \ldots, r$, and energy-momentum tensor $L_g(z)$ of the WZNW theory can be expressed in terms of Γ using the vertex operator representation [8, 9, 10]. To find operators which replace $J(z)$ and $L_g(z)$ in the constrained theory it is sufficient to quantize only the operators Γ.

Let F and G belong to the space Γ. According to Dirac's procedure we replace these operators by the operators $\tilde{F}$ and $\tilde{G}$ which satisfy the following commutator relation

$$[\tilde{F}, \tilde{G}] = [F, G] - \sum_{a,b} [F, \Phi_a][\Phi_a, \Phi_b]^{-1}[\Phi_b, G], \tag{13}$$

where $\Phi = (\Phi_a)$ are the constraints (12). It is easy to see that all the commutators in the right-hand side of (13) are c-functions. Therefore the new commutator is well-defined. Computations show that the nonvanishing commutators of the operators $\tilde{q}_s^j, \tilde{p}_s^j$ and $\tilde{a}_{sm}^j$ are given by

$$[\tilde{q}_A^j, \tilde{p}_A^l] = i\eta^{jl}, \qquad [\tilde{q}_I^j, \tilde{p}_I^l] = i\delta^{jl},$$

$$[\tilde{a}_{Am}^i, \tilde{a}_{An}^j] = m\eta^{ij}\delta_{m+n,0}, \qquad [\tilde{a}_{Im}^i, \tilde{a}_{In}^j] = m\delta^{ij}\delta_{m+n,0}, \tag{14}$$

where $I = d+1, \ldots, r$ and

$$\eta^{ij} = \begin{cases} \frac{(k-1)}{k} & \text{if } i = j, \\ -\frac{1}{k} & \text{if } i \neq j. \end{cases} \tag{15}$$

Note that $\eta^2 = \eta$. It follows from (14) that the constraints $\tilde{\Phi} = (\tilde{H}_{Ao}, \tilde{H}_{An}, \tilde{q}_A)$ commute with the operators $\tilde{\Gamma} = (\tilde{q}_s^j, \tilde{p}_s^j, \tilde{a}_{sm}^j)$

$$[\tilde{\Phi}, \tilde{\Gamma}] = 0 \tag{16}$$

The nonvanishing two-point functions of the fields

$$\tilde{\varphi}_s^j(z) = \tilde{q}_s^j - i\tilde{p}_s^j \log z + i \sum_{n \neq 0} \frac{\tilde{a}_{sn}^j}{n} z^{-n}, \tag{17}$$

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are read

$$< 0|\tilde{\varphi}_A^i(z)\tilde{\varphi}_A^j(w)|0 > = -\eta^{ij}\ln(z-w),$$
$$< 0|\tilde{\varphi}_I^i(z)\tilde{\varphi}_I^j(w)|0 > = -\delta^{ij}\ln(z-w), \tag{18}$$

where the vacuum vector is defined by

$$\tilde{p}_s^i|0 > = 0, \qquad \tilde{a}_{sn}^i|0 > = 0 \quad \text{for} \quad n > 0 \tag{19}$$

It follows from (18) that the fields $\tilde{\varphi}_s^j$ can be expressed as follows

$$\tilde{\varphi}_A^i = \eta^{ij}\varphi_A^j, \qquad \tilde{\varphi}_I^i = \varphi_I^i \tag{20}$$

Using the commutator relations (14) and the bosonic construction of the currents and Virasoro generator of the WZNW theory,one can compute operator product expansions of these operators in the constrained theory.

These results can be generalized to non-simply-laced algebras using the representation of the associated affine KM algebras in terms of the operators Γ and fermion fields[11, 12].

3.The energy-momentum tensor L_g of the WZNW theory for g simply-laced can by written as [10]

$$L_g(\varphi) = \frac{1}{2(k+h)}\{(1+h)\sum_{s=1}^{r}\sum_{j=1}^{k} : (i\partial_z\varphi_s^j)^2 : +$$

$$+2\sum_{s=1}^{r}\sum_{i<j}^{k} : (i\partial_z\varphi_s^j)(i\partial_z\varphi_s^j) : +2\sum_{\alpha}\sum_{i<j}^{k} : \exp[i\alpha\cdot(\varphi^i-\varphi^j)] : c_\alpha^i c_{-\alpha}^j\}, \tag{21}$$

where h is the dual Coxeter number of the algebra g, c_α^i are cocycles and the bosons φ_s^i satisfy eq.(5).

Let us consider this operator in the constrained theory where the fields φ_s^i are replaced by $\tilde{\varphi}_s^i$.Substituting (20) into (21), we get

$$L_g(\tilde{\varphi}) = L_g(\varphi) - L_{u(1)^d}(\varphi), \tag{22}$$

Thus the energy-momentum tensor of the constrained WZNW theory can be written in the coset $g/u(1)^d$ form.

Consider the case of $su(2)/u(1)$. $\hat{s}u(2)$ algebra is generated by the currents

$$E^+(z) = \sum_{j=1}^{k} : e^{i\sqrt{2}\varphi^j(z)} :, \qquad E^-(z) = \sum_{j=1}^{k} : e^{-i\sqrt{2}\varphi^j(z)} :, \tag{23}$$

where $\varphi^i \equiv \varphi_1^i$ satisfy eq.(5). Upon quantization of the system with the constrainted $\hat{u}(1)$ current

$$H(z) \equiv \sum_{j=1}^{k} i\partial_z\varphi^j(z) \approx 0, \tag{24}$$

the fields φ^i are replaced by

$$\tilde{\varphi}^i = \eta^{ij}\varphi^j. \tag{25}$$

Substituting (25) into (23) we get the parafermionic currents ψ_1^+ and ψ_1

$$\sqrt{k}\psi_1^+ = \sum_{j=1}^k : e^{i\sqrt{2}\eta^{jl}\varphi^l} :, \qquad \sqrt{k}\psi_1 = \sum_{j=1}^k : e^{-i\sqrt{2}\eta^{jl}\varphi^l} :. \tag{26}$$

Computations show that ψ_1^+ and ψ_1 satisfy the parafermionic algebra of ref.[4]:

$$\psi_1(z)\psi_1^+(w) = (z-w)^{-2+\frac{2}{k}}\left(I + \frac{k+2}{k}K_{su(2)}((w))(z-w)^2\right), \tag{27}$$

where $K_{su(2)}(z) = L_{su(2)}(\varphi(z)) - L_{u(1)}(\varphi(z))$. This proves the equivalence of the constrained $su(2)$ WZNW theory to the $su(2)/u(1)$ theory.

In conclusion using canonical quantization techniques we have quantized the WZNW theory invariant with respect to an affine KM algebra $\hat{g}$ with constrained $\hat{u}(1)^d$ currents.We have shown that Virasoro algebra of the constrained theory is the $g/u(1)^d$ coset Virasoro algebra.In the case of $su(2)/u(1)$ this correspondence also works for the currents.It seems likely that the $g/u(1)^d$ conformal field theory is equivalent to the constrained WZNW theory.It would be interesting to study the $\hat{g}$ WZNW theory with arbitrary constrained current algebra $\hat{h} \subset \hat{g}$ and compare it with the g/h conformal field theory.

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